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CORRELATION INEQUALITIES AND A CONJECTURE FOR PERMANENTS

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This paper presents conditions on nonnegative real valued functions $f_1, f_2, \ldots, f_m$ and $g_1, g_2, \ldots g_m$ implying an inequality of the type

$$\prod_{i=1}^{m} \int f_i(\mathbf{x}) d\mu(\mathbf{x}) \le \prod_{i=1}^{m} \int g_i(\mathbf{x}) d\mu(\mathbf{x}).$$

This "2m-function" theorem generalizes the "4-function" theorem of [2], which in turn generalizes a "2-function" theorem ([8]) and the celebrated FKG inequality. It also contains (and was partly inspired by) an "m against 2" inequality that was deduced in [5] from a general product theorem.

1. Introduction

Given two sequences $f_1, f_2, \ldots, f_m$ and $g_1, g_2, \ldots, g_m$ of nonnegative real valued functions, and a measure μ defined on a domain D, we are interested in determining conditions that guarantee the inequality:

$$\int_D f_1(\mathbf{x}) d\mu(\mathbf{x}) \ldots \int_D f_m(\mathbf{x}) d\mu(\mathbf{x}) \le \int_D g_1(\mathbf{x}) d\mu(\mathbf{x}) \ldots \int_D g_m(\mathbf{x}) d\mu(\mathbf{x}).$$

Our interest in this problem is motivated by the "4-function" theorem of Ahlswede and Daykin which provides such conditions in the case $m = 2$. This well-known result generalizes a number of important inequalities in combinatorics, probability theory and statistical mechanics including the FKG inequality, and related results due to Holley ([8]), Harris([7]), and Kleitman ([10]). Ahlswede and Daykin ([3]) developed a framework for inequalities on set functions based on a general product theorem which abstracts the induction step of the proof of the 4-function theorem. This was extended further by Daykin ([5]) who used this framework to deduce several corollaries including sufficient conditions on functions $f_1, f_2, \ldots, f_m$ and g_1, g_2 that imply:

$$\prod_{i=1}^{m} \int_D f_i(\mathbf{x}) d\mu(\mathbf{x}) \le \int_D g_1(\mathbf{x}) d\mu(\mathbf{x}) \int_D g_2(\mathbf{x}) d\mu(\mathbf{x}).$$

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in the case that $D=\{0,1\}^k$, and μ is the uniform probability measure ([5], Theorem 8).

In this paper, we formulate and prove a "$2m$-function" theorem that provides a natural generalization of the 4-function theorem. In addition it strengthens the result of Daykin mentioned above, settling some questions posed by Daykin.

We need the following notation and definitions: for $\mathbf{x}=(x_1,x_2,\ldots,x_k)$ and $\mathbf{y}=(y_1,y_2,\ldots,y_k)$ in $\mathbb{R}^k$, $\mathbf{x}\vee\mathbf{y}$ and $\mathbf{x}\wedge\mathbf{y}$ in $\mathbb{R}^k$ are defined to have coordinates $(\mathbf{x}\vee\mathbf{y})_j=\max(x_j,y_j)$ and $(\mathbf{x}\wedge\mathbf{y})_j=\min(x_j,y_j)$. Given $\mathbf{x^1},\mathbf{x^2},\ldots,\mathbf{x^m}$ from $\mathbb{R}^k$ set $\wedge_{\mathbf{i}=1}^{\mathbf{m}}\mathbf{x^i}=\mathbf{x^1}\wedge\mathbf{x^2}\wedge\ldots\wedge\mathbf{x^m}$ and analogously for $\vee$. Also, for any set $S\subseteq\{1,\ldots,m\}$, define $\mathbf{x}^S=\wedge_{\mathbf{i}\in S}\mathbf{x^i}$, and

$$\mathbf{x}^{[\mathbf{l}]}=\bigvee_{S:|S|=l}\mathbf{x}^S,\quad \mathbf{l}=1,\ldots,m.$$

Thus, $\mathbf{x}^{[\mathbf{1}]}=\vee_{\mathbf{i}=1}^{\mathbf{m}}\mathbf{x^i},\mathbf{x}^{[\mathbf{m}]}=\wedge_{\mathbf{i}=1}^{\mathbf{m}}\mathbf{x^i}$; in general the jth coordinate of $\mathbf{x}^{[\mathbf{l}]}$ has the lth largest value among $\mathbf{x^i}_j,\mathbf{i}=1,\ldots,m$.

Definition. A σ-finite (nonnegative) measure μ on $\mathbb{R}^k$ is said to be an *FKG measure* if μ has a density function ϕ with respect to some product measure $d\sigma$ on $\mathbb{R}^k$, (that is, $d\sigma(\mathbf{x})=\Pi_{j=1}^k d\sigma_j(x_j)$, and $d\mu(\mathbf{x})=\phi(\mathbf{x})d\sigma(\mathbf{x})$), where ϕ satisfies for all $\mathbf{x}=(x_1,x_2,\ldots,x_k)$ and $\mathbf{y}=(y_1,y_2,\ldots,y_k)$ in $\mathbb{R}^k$,

$$\phi(\mathbf{x})\phi(\mathbf{y})\leq\phi(\mathbf{x}\vee\mathbf{y})\phi(\mathbf{x}\wedge\mathbf{y}).\tag{1}$$

Condition (1) is referred to as *multivariate total positivity* of order 2 (MTP_2) in [9].

The definition of FKG measures includes both discrete and continuous cases. In combinatorial applications, the most interesting examples are counting measures on finite distributive lattices. The counting measure on such a lattice D is shown to satisfy the definition by embedding D in a finite lattice L obtained as a product of chains, taking $d\sigma$ to be the counting measure on L, and the function ϕ to be the indicator function of D. In other discrete applications ϕ is a probability function satisfying (1), e.g., for the Ising model in statistical mechanics, ϕ is a probability function on $D=\{-1,1\}^k$, of the form $\phi(\mathbf{x})=Z^{-1}\exp\{\sum J_{i,j}x_ix_j\}$, with $J_{i,j}\geq 0$. Natural continuous cases arise when $d\sigma$ is Lebesgue measure and ϕ is a probability density satisfying (1). Examples of such probability densities are given in [9] and references therein, and include the multivariate normal under certain conditions, and the case of order statistics.

The main result of this paper is:

Theorem 1.1. *Let $f_1,f_2,\ldots,f_m$ and $g_1,g_2,\ldots,g_m$ be nonnegative real valued functions defined on $\mathbb{R}^k$ that satisfy the following condition: for every sequence $\mathbf{x^1},\mathbf{x^2},\ldots,\mathbf{x^m}$ of elements from $\mathbb{R}^k$*

$$f_1(\mathbf{x^1})f_2(\mathbf{x^2})\ldots f_m(\mathbf{x^m})\leq g_1(\mathbf{x}^{[\mathbf{1}]})g_2(\mathbf{x}^{[\mathbf{2}]})\ldots g_m(\mathbf{x}^{[\mathbf{m}]}).\tag{2}$$

Then, for any FKG measure μ on $\mathbb{R}^k$:

$$\int_{\mathbb{R}^k} f_1(\mathbf{x})d\mu(\mathbf{x}) \ldots \int_{\mathbb{R}^k} f_m(\mathbf{x})d\mu(\mathbf{x}) \leq \int_{\mathbb{R}^k} g_1(\mathbf{x})d\mu(\mathbf{x}) \ldots \int_{\mathbb{R}^k} g_m(\mathbf{x})d\mu(\mathbf{x}). \tag{3}$$

Moreover, for any sublattice D of $\mathbb{R}^k$,

$$\prod_{i=1}^{m} \int_D f_i(\mathbf{x})d\mu(\mathbf{x}) \leq \prod_{i=1}^{m} \int_D g_i(\mathbf{x})d\mu(\mathbf{x}). \tag{4}$$

We are not concerned with issues of integrability in this paper and so we always assume that integrals are well-defined. The case $m = 2$ of this theorem is the 4-function theorem.

The proof of the theorem is by induction on the dimension k. The induction step mimics that in [2] (and also could be deduced from the product theorems in [3] and [5]), and it is the basis ($k=1$) case that is new. The proof of the basis step is related to the approach to the 4-function theorem presented in [9] and utilizes the relationship between the desired inequality and an inequality of matrix permanents. Our investigations suggested the following:

Conjecture 1.1. *Let A and B be $m \times m$ nonnegative matrices. Suppose that for any nondecreasing sequence $1 \leq i_1 \leq i_2 \leq \ldots \leq i_m \leq m$ of integers and any permutation π of $\{1,2,\ldots,m\}$:*

$$A_{i_{\pi(1)},1}A_{i_{\pi(2)},2}\ldots A_{i_{\pi(m)},m} \leq B_{i_1,1}B_{i_2,2}\ldots B_{i_m,m}. \tag{5}$$

Then

$$\mathbf{Per}(A) \leq \mathbf{Per}(B).$$

This conjecture was formulated as the key step in the proof of Theorem 1.1. Thus far, we have proved the conjecture for the cases $m = 2$ and $m = 3$ and the case that for some r between 1 and m, both A and B consist of r identical rows followed by $m - r$ identical rows. As it turns out, this latter case is sufficient for us to prove Theorem 1.1. Nevertheless, we believe the general conjecture is of independent interest. For one thing, it includes as a special case the following classical inequality:

Theorem 1.2. *(Muirhead 1903, see([12]) or ([13]). Let $\mathbf{a} = (a_1,a_2,\ldots,a_m), \mathbf{b} = (b_1,b_2,\ldots,b_m)$ and $\mathbf{x} = (x_1,x_2,\ldots,x_m)$ be nonincreasing sequences of real numbers with the $\mathbf{x}$ nonnegative. Let A and B be the $m \times m$ matrices defined by $A_{i,j} = x_i^{a_j}$ and $B_{i,j} = x_i^{b_j}$. If $\mathbf{a} \prec \mathbf{b}$ (where $\prec$ denotes the majorization order, i.e., $a_1 + a_2 + \ldots + a_j \leq b_1 + b_2 + \ldots + b_j$ for each $j \leq m$, with equality for $j = m$) then $\mathbf{Per}(A) \leq \mathbf{Per}(B)$.*

It is easy to verify that the matrices in Muirhead's inequality satisfy the hypothesis of conjecture 1.1 (use [13],3.H.2.c, p. 92; or just check it).

The remainder of this paper is organized as follows. In Section 2 we derive Theorem 1.1 from two lemmas and a special case of Conjecture 1.1, and explain the relevance of the full conjecture to our problem. In Section 3 we prove the required special case of Conjecture 1.1 along with some brief comments on the conjecture. Finally, in Section 4, we use the theorem to deduce a combinatorial inequality for families of sets.

While this paper was being refereed, we learned that R. Aharoni and U. Keich ([1]) independently discovered the main result of this paper and some refinements. We benefited from discussions with them concerning their work.

2. Proof of Theorem 1.1

In this section we show the following:

1. Conjecture 1.1, if true, would yield a natural proof of Theorem 1.1.
2. A special case of conjecture 1.1, given by Lemma 2.1, yields Theorem 1.1 in the case that the measure μ is concentrated on $\{0,1\}^k$. With additional embedding and approximation arguments, we obtain a complete proof of Theorem 1.1.

Lemma 2.1. *Conjecture 1.1 holds for the case that A is an $m\times m$ nonnegative matrix such that for some r between 1 and m, A consists of r identical rows followed by $m-r$ identical rows, and B has the same structure.*

The proof of this lemma will be given in Section 3. The next lemma shows that the conclusion of Theorem 1.1 for $k=1$ follows from an inequality which will later be interpreted as an inequality between permanents. As usual, we use S_m to denote the set of all permutations of $\{1,2,\ldots,m\}$.

Lemma 2.2. *Let f_i and g_i be real valued functions defined on $\mathbb{R}$, and let μ be a σ-finite measure on $\mathbb{R}$, satisfying*

$$\sum_{\pi\in S_m}\prod_{i=1}^{m} f_i(x_{\pi(i)}) \leq \sum_{\pi\in S_m}\prod_{i=1}^{m} g_i(x_{\pi(i)}) \tag{6}$$

for any x_i in the support of μ, $i=1,\ldots,m$. Then

$$\prod_{i=1}^{m}\int_{\mathbb{R}} f_i(x)d\mu(x) \leq \prod_{i=1}^{m}\int_{\mathbb{R}} g_i(x)d\mu(x).$$

Proof. Assume first that μ is a probability measure, then extend by normalization to any finite measure, and by standard approximation to any σ-finite measure. Consider $x, x_1, \ldots, x_m$ to be *iid* random variables having the distribution μ. Taking expectations in (6) we obtain $m!\Pi_{i=1}^{m}E_\mu[f_i(x)] \leq m!\Pi_{i=1}^{m}E_\mu[g_i(x)]$ and the required result follows.

We now prove Theorem 1.1 in the special case that the measure μ is a product measure concentrated on $\{0,1\}^k$. We proceed by induction on k, starting with $k=1$. Defining $A_{i,j}=f_j(x^i)$ and $B_{i,j}=g_j(x^i)$, $i,j=1,2,\ldots,m$, we have $\mathbf{Per}(A)=\sum_{\pi\in S_m}\prod_{i=1}^{m} f_i(x_{\pi(i)})$ and $\mathbf{Per}(B)=\sum_{\pi\in S_m}\prod_{i=1}^{m} g_i(x_{\pi(i)})$. Thus in view of Lemma

2.2, the case $k = 1$ of the theorem will follow if we show that $\mathbf{Per}(A) \leq \mathbf{Per}(B)$ where $x^1, x^2, \ldots, x^m$ is an arbitrary sequence with all x^i in the support of μ. By the invariance of permanents under column permutations we may, without loss of generality, assume that $x^1 \geq x^2 \geq \ldots \geq x^m$.

Since the support of μ is $\{0,1\}$, we may assume that for some r, the first r of the x_i's equal 1, and the rest are equal to 0. Thus each of the first r rows of B is equal to $(g_1(1), g_2(1), \ldots, g_m(1))$, and each of the remaining $m-r$ rows is equal to $(g_1(0), g_2(0), \ldots, g_m(0))$. A has a similar structure, which is the structure specified by Lemma 2.1. To verify that A and B satisfy the hypothesis (5), note that:

$$A_{i_{\pi(1)},1} A_{i_{\pi(2)},2}, \ldots A_{i_{\pi(m)},m} = f_1(x^{i_{\pi(1)}}) f_2(x^{i_{\pi(2)}}) \ldots f_m(x^{i_{\pi(m)}})$$

$$\leq g_1(x^{i_1}) g_2(x^{i_2}) \ldots g_m(x^{i_m}) = B_{i_1,1}, B_{i_2,2} \ldots B_{i_m,m}, \tag{7}$$

where the inequality follows from the hypothesis (2) of Theorem 1.1 applied to the sequence $x^{i_{\pi(1)}}, x^{i_{\pi(2)}}, \ldots, x^{i_{\pi(m)}}$. This completes the basis step $k = 1$.

We now proceed with the induction step of the proof. Recall that we are in the special case that μ is a product measure, i.e., $d\mu(\mathbf{x}) = \Pi_{j=1}^{k} d\mu_j(x_j)$. Fix $\mathbf{x}^{\mathbf{i}} = (x_1^i, x_2^i, \ldots, x_k^i), \mathbf{i} = 1, \ldots, m$, set $\tilde{\mathbf{x}}^{\mathbf{i}} = (x_1^i, x_2^i, \ldots, x_{k-1}^i)$, and define

$$\tilde{f}_i(x) = f_i(\tilde{\mathbf{x}}^{\mathbf{i}}, x), \quad \tilde{g}_i(x) = g_i(\tilde{\mathbf{x}}^{[\mathbf{i}]}, x).$$

Then $\tilde{f}_i(x)$ and $\tilde{g}_i(x)$ satisfy the hypothesis of Theorem 1.1 in the variable x for the case $k = 1$. To see this one needs only to overcome the notation and apply the definitions. The case $k = 1$ of Theorem 1.1 now implies

$$\prod_{i=1}^{m} \int \tilde{f}_i(x) d\mu_k(x) \leq \prod_{i=1}^{m} \int \tilde{g}_i(x) d\mu_k(x)$$

which is equivalent to

$$\prod_{i=1}^{m} \int f_i(\tilde{\mathbf{x}}^{\mathbf{i}}, x) d\mu_k(x) \leq \prod_{i=1}^{m} \int g_i(\tilde{\mathbf{x}}^{[\mathbf{i}]}, x) d\mu_k(x).$$

The latter inequality says that the "marginal" functions defined by $p_i(\tilde{\mathbf{x}}) = \int f_i(\tilde{\mathbf{x}}, x) d\mu_k(x)$ and $q_i(\tilde{\mathbf{x}}) = \int g_i(\tilde{\mathbf{x}}, x) d\mu_k(x)$ satisfy the hypothesis of Theorem 1.1 as functions of $\tilde{\mathbf{x}} \in \mathbb{R}^{k-1}$. Applying the induction hypothesis, we conclude

$$\prod_{i=1}^{m} \int p_i(\tilde{\mathbf{x}}) \Pi_{j=1}^{k-1} d\mu_j(x_j) \leq \prod_{i=1}^{m} \int q_i(\tilde{\mathbf{x}}) \Pi_{j=1}^{k-1} d\mu_j(x_j).$$

Finally, note that

$$\int p_i(\tilde{\mathbf{x}}) \Pi_{j=1}^{k-1} d\mu_j(x_j) = \int f_i(\mathbf{x}) d\mu(\mathbf{x}),$$

and similarly for q_i, completing the induction step and the proof of Theorem 1.1 for the case of product measures with support on $\{0,1\}^k$.

In order to extend the result to general FKG measures we need the following lemma, which is essentially a result of Lorentz, [11], see [13] p.156 for a convenient reference. The proof is rather easy and involves a finite number of applications of the FKG condition.

Lemma 2.3. *Let ϕ be a nonnegative function on $\mathbb{R}^k$ satisfying the FKG condition (1). Then for any $\mathbf{x^1}, \mathbf{x^2}, \ldots, \mathbf{x^m}$ from $\mathbb{R}^k$*

$$\phi(\mathbf{x^1})\phi(\mathbf{x^2})\ldots\phi(\mathbf{x^m}) \leq \phi(\mathbf{x^{[1]}})\phi(\mathbf{x^{[2]}})\ldots\phi(\mathbf{x^{[m]}}).$$

The extension of Theorem 1.1 to FKG measures is now obvious: let μ be an FKG measure with density ϕ with respect to a product measure $d\sigma$. Replace the functions f_i and g_i by $f_i\phi$ and $g_i\phi$ respectively, $i=1,2,\ldots,m$. By Lemma 2.3 the latter functions satisfy the hypothesis of Theorem 1.1. Applying the established case of Theorem 1.1 for product measures with support on $\{0,1\}^k$ to these functions and the product measure $d\sigma$ yields the desired extension. Moreover, since the indicator function I_D of any sublattice of $\mathbb{R}^k$ satisfies the FKG condition (1) as well, the statement of (4) follows for any sublattice D in the same manner.

So far, we have proved Theorem 1.1 for the special case of FKG measures concentrated on $\{0,1\}^k$. The proof used only the special case of Conjecture 1.1 given in Lemma 2.1 (which will be proved in the next section). If Conjecture 1.1 were known to be true in general, the same argument would provide a natural proof of Theorem 1.1 in full generality.

In the absence of a proof of Conjecture 1.1, we complete the proof of Theorem 1.1 by reducing it to the known special case. As these arguments are standard and tedious, they will be described briefly. First replace $\mathbb{R}^k$ by $[-a,a]^k$ for large a. Given an FKG measure having the density ϕ with respect to a product measure $d\sigma$, we approximate $d\sigma$ by a discrete product measure $d\sigma_D$ on a finite sublattice D of $[-a,a]^k$. We now embed the sublattice D into $\{0,1\}^n$, for a suitable value of n. It is well known that such an embedding exists, preserving the lattice operations, so that we can regard D as a sublattice of $\{0,1\}^n$, and apply Theorem 1.1 to the density defined by $\phi d\sigma_D$ with respect to the counting measure on $\{0,1\}^n$. A standard approximation argument is now all that is required to complete the proof of Theorem 1.1.

Remark. Observe that the present proof has the rather awkward feature that for a general measure μ, the proof of the theorem requires that we know the result on $\{0,1\}^n$ for all n, and thus even the case $k=1$ requires the induction part of the argument. For example, when $k=1$ and μ has finite support of cardinality s, we need an embedding into $\{0,1\}^{s-1}$. A proof of Conjecture 1.1 would eliminate this aspect of our proof, as well as the need to apply a limiting argument to obtain the result for measures with infinite support.

3. Proof of Lemma 2.1 and discussion of Conjecture 1.1

Before we prove Lemma 2.1, we need the following notation and elementary result. For $\mathbf{a}=(a_1,a_2,\ldots,a_n)$ in $\mathbb{R}^n$ let $\mathbf{a}^*$ denote the decreasing rearrangement of $\mathbf{a}$, i.e., a vector having the same components as $\mathbf{a}$, arranged in decreasing order. For $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ define $\mathbf{a} \prec_w \mathbf{b}$ (weak majorization) if $a_1^*+a_2^*+\ldots+a_j^* \leq b_1^*+b_2^*+\ldots+b_j^*$ for each $j \leq n$, (not requiring equality for $j=n$). We now quote a simple lemma which will be soon needed.

Lemma 3.1. *(See, e.g., [13], p. 64, 3.C.1.b.) For* $h:\mathbb{R}^n \to \mathbb{R}$ *convex and increasing,* $\mathbf{a} \prec_w \mathbf{b}$ *implies* $\Sigma_{i=1}^n h(a_i) \le \Sigma_{i=1}^n h(b_i)$.

Proof of Lemma 2.1. For $\pi \in S_m$ define $\alpha_\pi = \Pi_{i=1}^m A_{\pi(i),i}$, $\beta_\pi = \Pi_{i=1}^m B_{\pi(i),i}$. Then, $\mathbf{Per}(A) \le \mathbf{Per}(B)$ is equivalent to $\Sigma\alpha_\pi \le \Sigma\beta_\pi$. Applying Lemma 3.1 with $h(x) = e^x$, the latter inequality follows provided we show that $(log\alpha_{\pi_1},\dots,log\alpha_{\pi_{m!}}) \prec_w (log\beta_{\pi_1},\dots,log\beta_{\pi_{m!}})$. By definition of the partial ordering $\prec_w$ this relation is equivalent to the statement that for any $V \subseteq S_m$ there exists a set $W \subseteq S_m$ with $|V| = |W|$ and

$$\prod_{\pi\in V} \alpha_\pi \le \prod_{\pi\in W} \beta_\pi \quad . \tag{8}$$

Recall that A has only two distinct rows, $(s_1, s_2, \dots, s_m)$ and $(t_1, t_2, \dots, t_m)$, say. Then

$$\prod_{\pi\in V} \alpha_\pi = \prod_{j=1}^m s_j^{k_j} t_j^{l_j} \tag{9}$$

for some nonnegative integers satisfying $k_j + l_j = |V|, j = 1,\dots,m$. Denote the two distinct rows of B by $\mathbf{u} = (u_1, u_2, \dots, u_m)$ and $\mathbf{v} = (v_1, v_2, \dots, v_m)$ with $\mathbf{u}$ preceding $\mathbf{v}$. Let $\eta \in S_m$ be such that $k_{\eta(1)} \ge k_{\eta(2)} \ge \dots \ge k_{\eta(m)}$. We claim that the expression in (9) is less than or equal to

$$\prod_{j=1}^m u_j^{k_{\eta(j)}} v_j^{l_{\eta(j)}}. \tag{10}$$

To see this note first that in the present case condition (5) reduces to

$$s_{\pi(1)} \cdots s_{\pi(j)} t_{\pi(j+1)} \cdots t_{\pi(m)} \le u_1 \cdots u_j v_{j+1} \cdots v_m \tag{11}$$

for any $j = 0, 1, \dots, m$ and $\pi \in S_m$. Thus defining for $0 \le j \le m$, $\mathcal{S}_j = s_{\eta(1)} \cdots s_{\eta(j)} t_{\eta(j+1)} \cdots t_{\eta(m)}$ and $\mathcal{U}_j = u_1 \dots u_j v_{j+1} \cdots v_m$ we have $\mathcal{S}_j \le \mathcal{U}_j$ for all j. Setting $k_{\eta(0)} = |V|$, and $k_{\eta(m+1)} = 0$, we have

$$\prod_{\pi\in V} \alpha_\pi = \prod_{j=1}^m \mathcal{S}_j^{k_{\eta(j)} - k_{\eta(j+1)}} \le \prod_{j=1}^m \mathcal{U}_j^{k_{\eta(j)} - k_{\eta(j+1)}} = \prod_{j=1}^m u_j^{k_{\eta(j)}} v_j^{l_{\eta(j)}},$$

thus the desired inequality between the expressions in (9) and (10) holds. Finally note that the quantity in (10) equals $\prod_{\pi\in W} \beta_\pi$ for the coset $W = V\eta = \{\pi\eta : \pi \in V\}$. Thus (8) is established and the proof is complete.

Remark. For $m = 2$ or 3 the inequality (8) can be proved for all sets V by exhausting all possible sets, and Conjecture 1.1 follows. For $m \ge 4$ such an exhaustive proof becomes too exhausting. We have come up with ways of constructing analogs for (10) which exceed the corresponding analogs of (9), however, we do not have a general method for carrying out the last step of the proof, that is, for showing that such an expression arises as $\prod_{\pi\in W} \beta_\pi$ for a suitable $W \subseteq S_m$. The analog of $\binom{k_1 \dots k_m}{l_1 \dots l_m}$ in the general case is a doubly stochastic matrix times a constant, and Conjecture 1.1 appears to be related to various other interesting questions on such matrices, which we may pursue elsewhere.

4. A Combinatorial Inequality

One of the motivating examples for the 4-function theorem was the following combinatorial inequality due to Daykin. For set collections $\mathcal{A}$ and $\mathcal{B}$, define:

$$\mathcal{A} \vee \mathcal{B} = \{A \cup B \ : \ A \in \mathcal{A}, B \in \mathcal{B}\},$$

$$\mathcal{A} \wedge \mathcal{B} = \{A \cap B \ : \ A \in \mathcal{A}, B \in \mathcal{B}\}.$$

Daykin's inequality ([4]) asserts that for any two collections $\mathcal{A}$ and $\mathcal{B}$,

$$|\mathcal{A}||\mathcal{B}| \leq |\mathcal{A} \vee \mathcal{B}||\mathcal{A} \wedge \mathcal{B}|. \tag{12}$$

Theorem 1.1 yields the following generalization. Let $\mathcal{A}^1, \mathcal{A}^2, \dots, \mathcal{A}^m$ be set collections. For any set $S \subseteq \{1, \dots, m\}$, define $\mathcal{A}^S = \wedge_{i \in S} \mathcal{A}^i$, and

$$\mathcal{A}^{[l]} = \bigvee_{S:|S|=l} \mathcal{A}^S, \quad l = 1, \dots, m.$$

Theorem 4.1. *For any set collections $\mathcal{A}^1, \mathcal{A}^2, \dots, \mathcal{A}^m$,*

$$|\mathcal{A}^1||\mathcal{A}^2| \dots |\mathcal{A}^m| \leq |\mathcal{A}^{[1]}||\mathcal{A}^{[2]}| \dots |\mathcal{A}^{[m]}|.$$

For example, in the case that $m = 3$, this theorem asserts:

$$|\mathcal{A}^1||\mathcal{A}^2||\mathcal{A}^3|$$
$$\leq |\mathcal{A}^1 \vee \mathcal{A}^2 \vee \mathcal{A}^3||(\mathcal{A}^1 \wedge \mathcal{A}^2) \vee (\mathcal{A}^1 \wedge \mathcal{A}^3) \vee (\mathcal{A}^2 \wedge \mathcal{A}^3)||\mathcal{A}^1 \wedge \mathcal{A}^2 \wedge \mathcal{A}^3|.$$

Theorem 4.1 is obtained from Theorem 1.1 as follows. Let U denote the union of all of the sets in the collections $\mathcal{A}^i$. We can identify the lattice of subsets of U with the sublattice $\{0,1\}^{|U|}$ of $\mathbb{R}^{|U|}$. Taking f_i to be the indicator function of $\mathcal{A}^i$ and g_i to be the indicator function of $\mathcal{A}^{[i]}$ it is easy to verify that the hypothesis (2) is satisfied. Taking the measure $d\mu$ to be the counting measure on $\{0,1\}^{|U|}$, the conclusion of Theorem 4.1 is now equivalent to the conclusion (4) of Theorem 1.1.

Remark. It might be tempting to think that Theorem 4.1 can be deduced by repeated application of inequality (12). That this is not the case was pointed out to us by R. Aharoni and U. Keich.

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